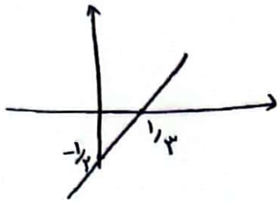
۱-

$y = \frac{3x-1}{2} = \frac{3}{2}x - \frac{1}{2}$

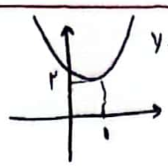


$S = \frac{\frac{1}{3} \times \frac{1}{2}}{2} = \frac{1}{12}$

برخورد با محور x: $\frac{3}{2}x - \frac{1}{2} = 0 \rightarrow x = \frac{1}{3}$

۲.

الف) $y = x^2 - 2x + 3 \;,\; x \in [1, +\infty)$

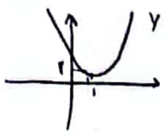


$\xrightarrow{x \geq 1}$

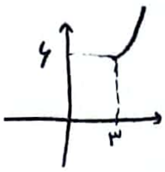
تابع یک به یک

$y = (x-1)^2 - 1 + 3 \Rightarrow \sqrt{y-2} = |x-1| \xrightarrow{x \geq 1} x = \sqrt{y-2} + 1 \rightarrow f^{-1}(x) = \sqrt{x-2} + 1 \quad D_{f^{-1}} = R_f \quad x \geq 2$

ب) $y = x^2 - 2x + 3 \;,\; x \in [3, +\infty)$



$\xrightarrow{x \geq 3}$



یک به یک

$y = (x-1)^2 - 1 + 3 \Rightarrow \sqrt{y-2} = |x-1| \xrightarrow{x \geq 3} x = \sqrt{y-2} + 1 \rightarrow f^{-1}(x) = \sqrt{x-2} + 1 \quad D_{f^{-1}} = R_f \quad x \geq 6$

۳-

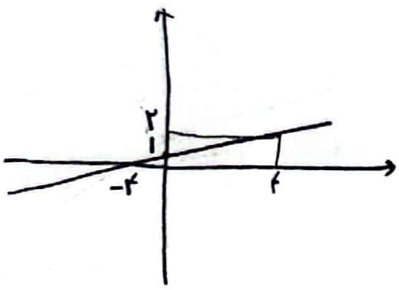
$f(x) = 2x - \sqrt{16 - 4x(4-x)} = 2x - \sqrt{4(4 - x(4-x))} = 2x - \sqrt{4(4 - 4x + x^2)} = 2x - \sqrt{4(x-2)^2}$

$= 2x - 2|x-2| = \begin{cases} 4 & x \geq 2 \\ 4x - 4 & x < 2 \end{cases}$

تابع در بازه‌ی $(-\infty, 2]$ یک به یک است.

$f^{-1}(x)$: $y = 4x - 4 \rightarrow x = \frac{y+4}{4}$

$\rightarrow f^{-1}(x) = \frac{x+4}{4} \quad D_{f^{-1}} = R_f \quad x < 4$



$S = \frac{2 \times 8}{2} = 8$

۴-

برای بررسی یک به یک بودن تابع : $f(x_1)=f(x_2) \to x_1=x_2$ $\qquad \dfrac{x_1}{\sqrt{x_1^2-5}} = \dfrac{x_2}{\sqrt{x_2^2-5}} \xrightarrow[\text{x_1 و x_2 هم علامت اند [برای مساوی شدن] } x_1 \cdot x_2>0]{\text{① مخرج هردو مثبت پس}}$

$$x_1\sqrt{x_2^2-5} = x_2\sqrt{x_1^2-5} \xrightarrow{()^2} x_1^2(x_2^2-5) = x_2^2(x_1^2-5) \to \cancel{(x_1x_2)^2} - 5x_1^2 = \cancel{(x_1x_2)^2} - 5x_2^2$$

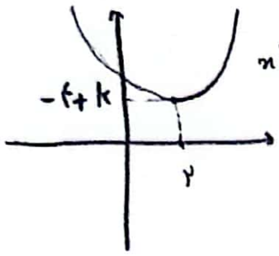
$$\to 5x_1^2 = 5x_2^2 \to x_1 = \pm x_2 \xrightarrow[x_1x_2>0]{①} x_1 = x_2$$ یک به یک است.

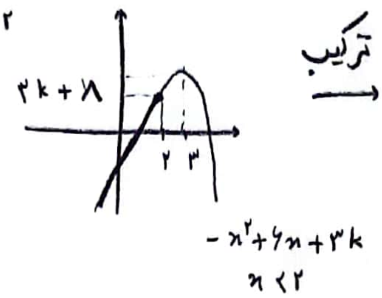
$$y = \frac{x}{\sqrt{x^2-5}} \to y\sqrt{x^2-5} = x \xrightarrow{()^2} y^2(x^2-5) = x^2 \to y^2x^2 - 5y^2 = x^2 \to y^2x^2 - x^2 = 5y^2$$

$$\to x^2 = \frac{5y^2}{y^2-1} \to x = \sqrt{\frac{5y^2}{y^2-1}} \to f^{-1}(x) = \sqrt{\frac{5x^2}{x^2-1}}$$

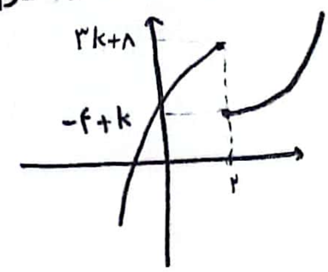
۵- برای حل این مسئله دو حالت در نظر می‌گیریم ، $k>4$ و $k<4$ (در حل متوجه می‌شویم چرا؟)

$\xrightarrow{k>4}$



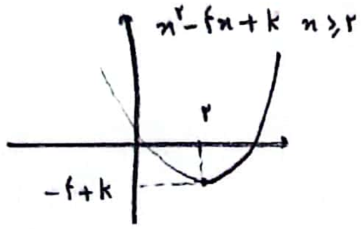


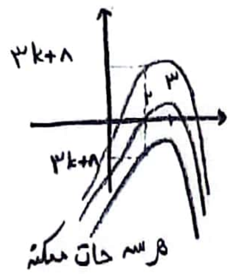
$\xrightarrow{\text{ترکیب}}$



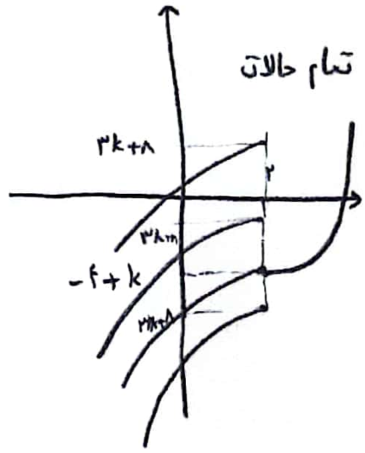
یک به یک نمی باشد

$\xrightarrow{k<4}$





$\xrightarrow{\text{ترکیب}}$



$\xrightarrow[\text{یک به یک بودن}]{\text{برای}}$ $3k+8 < -4+k \to 2k < -12 \to \boxed{k<-6}$

-۶

$$f(x)=3x+|x| \rightarrow \begin{cases} 4x & x\geq 0 \\ 2x & x<0 \end{cases} \xrightarrow{f^{-1}(x)} \begin{cases} \frac{x}{4} & x\geq 0 \\ \frac{x}{2} & x<0 \end{cases} \rightarrow f^{-1}(x)=\begin{cases} \frac{x}{4} & x\geq 0 \\ \frac{x}{2} & x<0 \end{cases}$$

$$g(x)=f^{-1}(x)=ax+b|x| \begin{cases} x\geq 0: & ax+bx=\frac{x}{4} \rightarrow a+b=\frac{1}{4} \\ x<0: & ax-bx=\frac{x}{2} \rightarrow a-b=\frac{1}{2} \end{cases} \rightarrow a=\frac{3}{8},\ b=\frac{-1}{8}$$

$$\rightarrow ab=\frac{-3}{64}$$

-۷

$$f(x)=\frac{3x+4}{x+m} \xrightarrow[\text{عبور}]{(2,-10)} \frac{6+4}{2+m}=-10 \rightarrow 2+m=-1 \rightarrow m=-3$$

$$f(x)=\frac{3x+4}{x-3} \xrightarrow{a+d=0} f(x)=f^{-1}(x)$$

$$y=\underbrace{f\circ f^{-1}(x)}_{x}+f^{-1}(x)=x+f^{-1}(x)=x+\frac{3x+4}{x-3} \xrightarrow[\text{تقاطع می‌دهیم}]{y=x \text{ با}} \frac{3x+4}{x-3}+x=x \rightarrow \frac{3x+4}{x-3}=0$$

$$\rightarrow x=\boxed{\frac{-4}{3}}$$

-۸

$$f(3-2x)=2-g\left(\frac{x}{2}\right)\ ,\quad 4g^{-1}(4)+f^{-1}(-2)=?$$

$$f(3-2x_1)=-2 \rightarrow f^{-1}(-2)=3-2x_1 \quad ①$$

$$\xrightarrow[\text{معادله}]{\text{در } x_1} f(3-2x_1)=2-g\left(\frac{x_1}{2}\right) \rightarrow g\left(\frac{x_1}{2}\right)=4 \rightarrow g^{-1}(4)=\frac{x_1}{2} \quad ②$$

$$\xrightarrow{①+②} 3-2x_1+2x_1=\boxed{3}$$

۹-

$$y=\frac{5x-13}{1-x}\xrightarrow[\text{(۰,۰)}]{\text{قرینه نسبت به نقطه}}-y=\frac{-5x-13}{1+x}\rightarrow y=\frac{5x+13}{1+x}\rightarrow y=\frac{5(x-2)+13}{1+x-2}-3$$

$$=\frac{5x+3}{x-1}-3=\frac{5x+3-3x+3}{x-1}=\frac{2x+6}{x-1}=f(x)$$

$$f^{-1}(x):\ y=\frac{2x+6}{x-1}\rightarrow yx-y=2x+6\rightarrow yx-2x=y+6\rightarrow f^{-1}(x)=\frac{x+6}{x-2}$$

$$f(x)=f^{-1}(x)\rightarrow\frac{2x+6}{x-1}=\frac{x+6}{x-2}\rightarrow(2x+6)(x-2)=(x-1)(x+6)\rightarrow 2x^2+2x-12=x^2+5x-6$$

$$\rightarrow x^2-3x-6=0\rightarrow S=x_1+x_2=\frac{-b}{a}=\boxed{3}$$

۱۰-

$$f(x)=x+4[x]$$

$$f(x)-f^{-1}(x)=?\quad D_{f\cap f^{-1}}=D_f\cap R_f=\mathbb{R}\cap\{\dots,[-5,-4)\cup[0,1)\cup[5,6)\dots\}=\dots,[-5,-4)\cup[0,1)\cup[5$$

$$f(x)=\begin{cases}\vdots\\ x-20 & -5\le x<-4\\ x & 0\le x<1\\ x+20 & 5\le x<6\\ \vdots\end{cases},\quad f^{-1}(x)=\begin{cases}-25\le x+4[x]<-24 & -5\le x<-4\\ 0\le x+4[x]<1 & 0\le x<1\\ 25\le x+4[x]<26 & 5\le x<6\end{cases}\rightarrow\begin{cases}x+4 & -5\le x<-4\\ x & 0\le x<1\\ x-4 & 5\le x<6\end{cases}$$

$$\xrightarrow{(-)}\begin{cases}-24 & -5\le x<-4\\ 0 & 0\le x<1\\ 24 & 5\le x<6\\ \vdots\end{cases}$$