بنام خدا

کیوان مکارم منش — تکلیف شماره ۱۷

۱)

می دانیم $\tan^2 x + 1 = \frac{1}{\cos^2 x} \to 8\cos x - \frac{1}{\cos^2 x} + \cancel{1} = \cancel{1}$

$\to 8\cos x = \frac{1}{\cos^2 x} \xrightarrow[\times \cos^2 x]{\cos x \neq 0} 8\cos^3 x = 1 \to \cos^3 x = \frac{1}{8} \xrightarrow{\sqrt[3]{}} \cos x = \frac{1}{2}$

$\to \cos x = \frac{1}{2} = \cos(\frac{\pi}{3}) \to x = 2k\pi + \frac{\pi}{3} \to k = 0 \Rightarrow x = \{\frac{\pi}{3}\}$ ✓

$x = 2k\pi - \frac{\pi}{3} \xrightarrow{x \in [0, 2\pi]} k = 1 \Rightarrow x = \{2\pi - \frac{\pi}{3}\}$ ✓

درنتیجه ۲ جواب $\{\frac{\pi}{3}, \frac{5\pi}{3}\}$ را داخل بازه داریم.

۲)

$5(1 - \cos^2 x) + 2(4\cos^3 x - 3\cos x) = -2 \to 5 - 5\cos^2 x + 8\cos^3 x - 6\cos x = -2$

$\xrightarrow{\cos x = t} 8t^3 - 5t^2 - 6t + 7 = 0 \xrightarrow{\text{تجزیه}} (t+1)(\underbrace{8t^2 - 13t + 7}_{\Delta < 0}) = 0 \to t = -1$

$\cos x = -1 \to x = 2k\pi + \pi$

$\xrightarrow{x \in [-\pi, \pi]} x = \{\pi, -\pi\} \to$ ۲ جواب داریم ✓

۳)

$2\sin x(\cos 2x) + \sin x = 1 \xrightarrow{\cos 2x = 1 - 2\sin^2 x} 2\sin x(1 - 2\sin^2 x) + \sin x = 1$

$\xrightarrow{\sin x = t} 2t - 4t^3 + t = 1 \to 4t^3 - 3t + 1 = 0 \xrightarrow[\text{بخش پذیر است}]{t+1 \text{ بر}}$

$\to (t+1)(4t^2 - 4t + 1) = 0 \xrightarrow{t+1=0}$ ① $\sin x = -1 \to x = 2k\pi + \frac{3\pi}{2}$

② $4t^2 - 4t + 1 = 0 \to t = \frac{1}{2} \to$ ② $\sin x = \frac{1}{2} \to x = 2k\pi + \frac{\pi}{6}$

$x = 2k\pi + \frac{5\pi}{6}$

$\xrightarrow{x \in [0, 2\pi]} x = \{\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\} \xrightarrow{\text{مجموع}} \frac{\pi}{6} + \frac{5\pi}{6} + \frac{3\pi}{2} = \pi + \frac{3\pi}{2} = \frac{5\pi}{2}$ ✓

۴)

$\frac{1}{\sin(\frac{\pi}{2} + 2x)} + \frac{1}{\cos(\frac{\pi}{2} + 4x)} = 0 \to \frac{1}{\cos 2x} + \frac{-1}{\underbrace{\sin 4x}_{2\sin 2x \cos 2x}} = 0$

$= \frac{2\sin 2x - 1}{2\sin 2x \cos 2x} = 0 \Rightarrow \frac{2\sin 2x - 1}{\sin 4x} = 0 \to 2\sin 2x - 1 = 0$

$\to \sin 2x = \frac{1}{2} \to 2x = 2k\pi + \frac{\pi}{6} \xrightarrow{\div 2} x = k\pi + \frac{\pi}{12}$ ① $\xrightarrow{x \in [0, \pi]} x = \{\frac{\pi}{12}, \frac{5\pi}{12}\}$

$2x = 2k\pi + \frac{5\pi}{6} \xrightarrow{\div 2} x = k\pi + \frac{5\pi}{12}$ ②

اختلاف جواب ها $\to \frac{5\pi}{12} - \frac{\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3} = \alpha \to \tan(2\alpha) = \tan(\frac{2\pi}{3}) = -\sqrt{3}$ ✓

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$\cos(x+\frac{\pi}{4}) = \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = \frac{1}{\sqrt{6}} \Rightarrow \frac{1}{\sqrt{2}}(\cos x - \sin x) = \frac{1}{\sqrt{3}}$

$\rightarrow \cos x - \sin x = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ (*)

$m(\cos x - \sin x) - 3\sqrt{6}\sin 2x = \sqrt{6} \xrightarrow{*} m(\frac{\sqrt{6}}{3}) - 3\sqrt{6}\sin 2x = \sqrt{6}$

$\xrightarrow{\times\sqrt{6}} 2m - 18\sin 2x = 6$

$\cos x - \sin x = \frac{\sqrt{6}}{3} \xrightarrow{\text{توان ۲}} \sin^2 x + \cos^2 x - \sin 2x = \frac{2}{3} \rightarrow 1 - \frac{2}{3} = \sin 2x \rightarrow \sin 2x = \frac{1}{3}$ (**)

$\xrightarrow{**} 2m - 18(\frac{1}{3}) = 6 \rightarrow 2m - 6 = 6 \rightarrow 2m = 12 \rightarrow m = 6$

۶

می‌دانیم: $\cos(\alpha) = \cos(-\alpha) \rightarrow \cos(x - \frac{\pi}{3}) = \cos(+\frac{\pi}{3} - x)$

$\rightarrow \underbrace{\sin(x+\frac{\pi}{6})}_{\theta}\underbrace{\cos(\frac{\pi}{3} - x)}_{\beta} = 1 \xrightarrow[\text{طبق متمم بودن زوایا}]{\beta + \theta = 90^\circ} \sin\theta = \cos\beta$

$\rightarrow \cos(x-\frac{\pi}{3})\cos(x-\frac{\pi}{3}) = 1 \rightarrow \cos^2(x-\frac{\pi}{3}) = 1 \rightarrow \cos(x-\frac{\pi}{3}) = \pm 1$

① $\cos(x-\frac{\pi}{3}) = 1 \rightarrow x - \frac{\pi}{3} = 2k\pi \rightarrow x = 2k\pi + \frac{\pi}{3}$ ✓

② $\cos(x-\frac{\pi}{3}) = -1 \rightarrow x - \frac{\pi}{3} = 2k\pi + \pi \rightarrow x = 2k\pi + \frac{4\pi}{3}$ ✓

$x \in [0, 2\pi]$ $\quad x = \{\frac{\pi}{3}, \frac{4\pi}{3}\} \rightarrow$ ۲ جواب دارد

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$\sin x + \sqrt{3}\cos x = \sqrt{2} \xrightarrow{\div 2} \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$

$\rightarrow \cos(\frac{\pi}{3})\sin x + \sin(\frac{\pi}{3})\cos x = \frac{\sqrt{2}}{2} \xrightarrow{\text{بسط } \sin(\alpha+\beta)} \sin(x + \frac{\pi}{3}) = \frac{\sqrt{2}}{2} = \sin(\frac{\pi}{4})$

① $x + \frac{\pi}{3} = 2k\pi + \frac{\pi}{4} \rightarrow x = 2k\pi - \frac{\pi}{12}$ ✓ $\xrightarrow{x \in [-\pi, 2\pi]} x = \{2\pi - \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{12}\}$

② $x + \frac{\pi}{3} = 2k\pi + \underbrace{\pi - \frac{\pi}{4}}_{\frac{3\pi}{4}} \rightarrow x = 2k\pi + \frac{5\pi}{12}$ ✓

$\rightarrow$ ۳ جواب دارد

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می‌دانیم: $\sin(\frac{3\pi}{2} - x) = -\cos x$

$\rightarrow 4\sin x(-\cos x) = 1 \rightarrow -4\cos x \sin x = 1 \rightarrow -2(2\sin x\cos x) = 1$

$\xrightarrow[\text{می‌دانیم}]{\sin 2\alpha = 2\sin\alpha\cos\alpha} -2(\sin 2x) = 1 \rightarrow \sin 2x = -\frac{1}{2} = \sin(\frac{7\pi}{6})$

① $2x = 2k\pi + \frac{7\pi}{6} \rightarrow x = k\pi + \frac{7\pi}{12}$ ✓ $\xrightarrow{x \in [0, 2\pi]} x = \{\frac{7\pi}{12}, \pi + \frac{7\pi}{12}, \pi - \frac{\pi}{12}, 2\pi - \frac{\pi}{12}\}$

② $2x = 2k\pi + \underbrace{\pi - \frac{7\pi}{6}}_{-\frac{\pi}{6}} \rightarrow x = k\pi - \frac{\pi}{12}$ ✓

$\xrightarrow{\text{جمع جواب‌ها}} \frac{7\pi}{12} + \pi + \frac{7\pi}{12} + \pi - \frac{\pi}{12} + 2\pi - \frac{\pi}{12} = 4\pi + \pi = \boxed{5\pi}$ ✓

۹

می‌دانیم: $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2} \Rightarrow (1+\cos 2\alpha)(1+\cos 4\alpha)(1+\cos 8\alpha) = \dfrac{1}{8}$

$\rightarrow (2\cos^2 \alpha)(2\cos^2 2\alpha)(2\cos^2 4\alpha) = \dfrac{1}{8} \rightarrow 8(\cos\alpha \cos 2\alpha \cos 4\alpha)^2 = \dfrac{1}{8}$

$\rightarrow (\cos\alpha \cos 2\alpha \cos 4\alpha)^2 = \dfrac{1}{64} \xrightarrow{\sqrt{\ }} \cos\alpha \cos 2\alpha \cos 4\alpha = \pm\dfrac{1}{8}$

$\xrightarrow{\frac{2\sin\alpha}{2\sin\alpha}} \dfrac{2\sin\alpha \cos\alpha \cos 2\alpha \cos 4\alpha}{2\sin\alpha} = \pm\dfrac{1}{8} \xrightarrow{\text{اتحاد}} \dfrac{\sin 2\alpha \cos 2\alpha \cos 4\alpha}{2\sin\alpha} = \pm\dfrac{1}{8}$

$\xrightarrow{\times\frac{2}{2}} \dfrac{2\sin 2\alpha \cos 2\alpha \cos 4\alpha}{4\sin\alpha} = \pm\dfrac{1}{8} \xrightarrow{\text{اتحاد}} \dfrac{\sin 4\alpha \cos 4\alpha}{4\sin\alpha} = \pm\dfrac{1}{8}$

$\xrightarrow{\times\frac{2}{2}} \dfrac{2\sin 4\alpha \cos 4\alpha}{8\sin\alpha} = \pm\dfrac{1}{8} \xrightarrow{\text{اتحاد}} \dfrac{\sin 8\alpha}{\sin\alpha} = \pm 1$ (I)

$\xrightarrow{①} \dfrac{\sin 8\alpha}{\sin\alpha} = 1 \xrightarrow{\sin\alpha \neq 0} \sin 8\alpha = \sin\alpha \rightarrow 8\alpha = 2k\pi + \alpha \rightarrow \alpha = \dfrac{2k\pi}{7}$ ✓

$8\alpha = 2k\pi + \pi - \alpha \rightarrow \alpha = \dfrac{2k\pi}{9} + \dfrac{\pi}{9}$ ✓ (II)

$② \rightarrow \dfrac{\sin 8\alpha}{\sin\alpha} = -1 \rightarrow \sin 8\alpha = -\sin\alpha \xrightarrow{\sin(\pi+\alpha) = -\sin\alpha} \sin 8\alpha = \sin(\pi+\alpha)$

$8\alpha = 2k\pi + \pi + \alpha \rightarrow \alpha = \dfrac{2k\pi}{7} + \dfrac{\pi}{7}$ ✓ (III)

$8\alpha = 2k\pi + \cancel{\pi} - \cancel{\pi} - \alpha \rightarrow \alpha = \dfrac{2k\pi}{9}$ ✓ (IV)

$\xrightarrow[\text{(IV)}]{\text{(I),(II),(III)}} x \in [0,\pi] \overset{A}{\Rightarrow} x = \left\{0, \dfrac{2\pi}{7}, \dfrac{4\pi}{7}, \dfrac{6\pi}{7}, \dfrac{\pi}{9}, \dfrac{\pi}{3}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \pi, \dfrac{\pi}{7}, \dfrac{3\pi}{7}, \dfrac{5\pi}{7}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{2\pi}{3}, \dfrac{8\pi}{9}\right\}$

$\rightarrow$ ماکزیمم جواب معادله : ۱۴ ✓

۱۰

$\tan 3x \cdot \tan x = 1 \rightarrow \tan 3x = \dfrac{1}{\tan x} = \cot x \xrightarrow{\cot x = \tan(\frac{\pi}{2} - x)}$

$\rightarrow \tan 3x = \tan\left(\dfrac{\pi}{2} - x\right) \rightarrow 3x = k\pi + \dfrac{\pi}{2} - x$

$\rightarrow 4x = k\pi + \dfrac{\pi}{2} \rightarrow x = \dfrac{k\pi}{4} + \dfrac{\pi}{8}$

$\xrightarrow{x \in [\pi, 2\pi]} x = \left\{\pi + \dfrac{\pi}{8}, \dfrac{5\pi}{4} + \dfrac{\pi}{8}, \dfrac{3\pi}{2} + \dfrac{\pi}{8}\right\} \xrightarrow[\text{ریشه‌ها}]{\text{جمع}} \pi + \dfrac{\pi}{8} + \dfrac{5\pi}{4} + \dfrac{\pi}{8} + \dfrac{3\pi}{2} + \dfrac{\pi}{8}$

$= \dfrac{33\pi}{8}$ ✓