«به نام خدا»

«کیوان مکارم شب ۲۲ — تکلیف شماره ۲۲

(۱)

شرط پیوسته بودن: $f_+(1) = f_-(1) = f(1) \to a+1 = b$ (1)

$$f(x) = \begin{cases} a+|x| & ; x<1 \\ b\sqrt[3]{x^2} & ; x \geq 1 \end{cases} \to f(x) = \begin{cases} a+x & ; 0<x<1 \\ a-x & ; x<0 \\ b\sqrt[3]{x^2} & ; x \geq 1 \end{cases} \to f'(x) = \begin{cases} 1 & 0<x<1 \\ -1 & x<0 \\ \frac{2b}{3\sqrt[3]{x}} & x>1 \end{cases}$$

← طبق ضابطه $f(x)$، تابع در $x \to 0$ و $x \to 0^+$، مشتق ناپذیر است، درنتیجه تابع نوشته شده $x \to 1^+$ و $x \to 1^-$، مشتق های برابری داشته باشد.

$$\to \frac{2b}{3\sqrt[3]{1}} = 1 \to \boxed{b = \frac{3}{2}} \xrightarrow{(1)} a = \frac{3}{2} - 1 = \frac{1}{2} \to \boxed{a = \frac{1}{2}}$$

$$\to a+b = \frac{3}{2} + \frac{1}{2} = 2 \checkmark$$

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$f(1) = 2 \qquad |x-2| = -x+2$

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = f'(1) \to \lim_{x \to 1} \frac{2|x-2| - 2}{x-1} = \lim_{x \to 1} \frac{2(-x+2-1)}{x-1}$$

$$\to \lim_{x \to 1} \frac{2(1-x)}{x-1} = -2 = f'(1) \quad (1)$$

می دانیم: $\sqrt{x^2 - 4x + 4} = \sqrt{(x-2)^2} = |x-2|$

$$g(x) = \frac{|x-2| + \sqrt{3x}}{(\sqrt[3]{x^2})} \Rightarrow g(x) = \begin{cases} \frac{x-2+\sqrt{3x}}{(\sqrt[3]{x^2})} & x>2 \\ \frac{2-x+\sqrt{3x}}{(\sqrt[3]{x^2})} & 0<x<2 \end{cases}$$

$$g'(x) = \begin{cases} \frac{\left(1+\frac{3}{2\sqrt{3x}}\right)(\sqrt[3]{x^2}) - \left(\frac{2}{3\sqrt[3]{x}}\right)(x-2+\sqrt{3x})}{(\sqrt[3]{x^2})^2} & x>2 \\ \frac{\left(-1+\frac{3}{2\sqrt{3x}}\right)(\sqrt[3]{x^2}) - \left(\frac{2}{3\sqrt[3]{x}}\right)(-x+2+\sqrt{3x})}{(\sqrt[3]{x^2})^2} & 0<x<2 \end{cases} \to g'(1) = \frac{\left(-1+\frac{3}{2\sqrt{3}}\right)(1) - \left(\frac{2}{3}\right)(1+\sqrt{3})}{1}$$

$$\to \frac{\sqrt{3}}{2} - 1 - \frac{2}{3} - \frac{2\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} - \frac{5}{3} \to f'(1) - 2g'(1) = -2 + \sqrt{3} + \frac{10}{3}$$

$$\to \boxed{\sqrt{3} + \frac{4}{3}} \checkmark$$

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$$f(x) = \begin{cases} bx+c & x<a \\ \frac{1}{x} & x \geq a \end{cases} \to f'(x) = \begin{cases} b & x<a \\ -\frac{1}{x^2} & x>a \end{cases}$$

$$\to f_+(a) = f_-(a) \Rightarrow ab + c = \frac{1}{a} \; (1) \quad , \quad f'_+(a) = f'_-(a) \Rightarrow b = -\frac{1}{a^2} \; (2)$$

$$(1),(2) \to a\left(\frac{-1}{a^2}\right) + c = \frac{1}{a} \to \frac{-1}{a} + c = \frac{1}{a} \to \frac{2}{a} = c \to \boxed{ac = 2}$$

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اولاً باید پیوسته باشد $\Rightarrow f(a^+) = f(a^-) \Rightarrow b = b(a-a)^m \Rightarrow b=b$ ✓

ثانیاً مشتق چپ و راست باید متفاوت باشد. $\Rightarrow f'(x) = \begin{cases} 0 & x<a \\ m(x-a)^{m-1} & x>a \end{cases} \Rightarrow$

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می دانیم: $(fog)'(x) = g'(x)\, f'(g(x)) \Rightarrow g'(-\sqrt[3]{2})\, f'(g(-\sqrt[3]{2})) =$

$= \left(f(g(-\sqrt[3]{2}))\right)' \Rightarrow fog(x) = \dfrac{1}{\sqrt[3]{x^3-|x^3| - |x^3-|x^3||}}$

$fog(x) = \begin{cases} x^3 - x^3 - 0 = 0 & x>0 \\ \dfrac{x^3+x^3-2|x^3|}{4|x^3|} & x<0 \end{cases} \Rightarrow fog(x) = \begin{cases} 0 & x>0 \\ 4x^3 & x<0 \end{cases}$

$\Rightarrow (fog)'(x) = \begin{cases} 0 & x>0 \\ 12x^2 & x<0 \end{cases} \Rightarrow fog'(-\sqrt[3]{2}) = 12(-\sqrt[3]{2})^2 = 12\sqrt[3]{4}$ ✓

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$f(x) = -ax-5 \Rightarrow f(3) = -3a-5$

$f'(x) = -a \Rightarrow f'(3) = -a \Rightarrow -a-(-3a-5) = 2$

$\Rightarrow -a+3a+5 = 2 \Rightarrow 2a = -3 \Rightarrow a = -\frac{3}{2} \Rightarrow f'(3) = \frac{3}{2}$ ✓

$f(3) = \frac{3}{2}(3) - 5 = \frac{9}{2} - \frac{10}{2} = \frac{-1}{2}$ ✓

$\Rightarrow \dfrac{f(3)}{f'(3)} = \dfrac{-\frac{1}{2}}{\frac{3}{2}} = -\frac{1}{3}$ ✓

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$(fog(x))' = g'(x)\, f'(g(x))$

$g\left(\frac{3}{\sqrt{8}}\right) = \dfrac{1}{\sqrt[3]{\frac{9}{8}-1}} = \dfrac{1}{\sqrt[3]{\frac{1}{8}}} = 2 \;,\; g'(x) = \dfrac{\dfrac{-2x}{3\sqrt[3]{(x^2-1)^2}}}{\left(\sqrt[3]{x^2-1}\right)^2} \Rightarrow g'\left(\frac{3}{\sqrt{8}}\right) = -8\sqrt{2}$

$g\left(\frac{3}{\sqrt{8}}\right) = 2 \Rightarrow f'(2) \Rightarrow \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2} \dfrac{\left(2\left[x^2+\frac{1}{2}\right]\right)^2 + 1 - 65}{x-2} = \lim\limits_{x\to 2} \dfrac{16(x^2-4)}{x-2}$

$= \lim\limits_{x\to 2} 16(x+2) = 64 \Rightarrow (fog)'\left(\frac{3}{\sqrt{8}}\right) = -8\sqrt{2} \times 64 = -512\sqrt{2}$

$\Rightarrow \dfrac{-512\sqrt{2}}{-128\sqrt{2}} = 4$ ✓

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$f(x \pm nT) = f(x) \Rightarrow f(3x+2(5)) = f(3x+5)$

$g(x) = f(x+1) + f(3x+5) \Rightarrow g'(x) = f'(x+1) + 3f'(3x+5) \Rightarrow g'(-2) = f'(-1) + 3f'(-1)$

$\Rightarrow g'(-2) = 4f'(-1) = 4\left(\frac{3}{2}\right) = 6$ ✓

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$$\lim_{h\to 0}\frac{f^2(5-h)-3f(5-h)+2}{h(5-h)} = \lim_{h\to 0}\frac{(f(5-h)-1)(f(5-h)-2)}{h(5-h)} \quad ①$$

$$\to f(5) = (5-4)\sqrt[3]{8} = 2 \xrightarrow{①} \lim_{h\to 0}\frac{(f(5-h)-1)}{(5-h)} \times \frac{(f(5-h)-f(5))}{h} \to -f'(5)$$

$$\to \lim_{h\to 0}\frac{(f(5-h)-1)}{5-h} \times f'(5) = \lim_{h\to 0}\frac{f(5)-1}{5} \times f'(5) = ?$$

$$\to f'(x) = \frac{1}{3\sqrt[3]{(x+3)^2}} \times (x-4) + \sqrt[3]{x+3} \Rightarrow f'(5) = \frac{1}{3\times 4} + 2 = 2 + \frac{1}{12} = \frac{25}{12}$$

$$\to \lim_{h\to 0}\frac{1}{5} \times \frac{25}{12} = \frac{5}{12} \checkmark$$

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$$y = \frac{1}{2}x + \frac{1}{4} \xrightarrow{\text{خط عمود}} m' = -2 \to$$

$m = \frac{1}{2}$

شیب مماس تابع در $x=a$ ، باید -2 باشه

$$\to f'(x) = 2x - 4 \longrightarrow 2x - 4 = -2 \longrightarrow 2x = 2 \longrightarrow \boxed{x = 1 = a}$$

$$\to \text{نقطه مورد نظر} = \begin{bmatrix} 1 \\ f(1) \end{bmatrix} \longrightarrow f(1) = 1 - 4 + 5 = 2 \to \begin{bmatrix} 1 \\ 2 \end{bmatrix} \checkmark$$