Subject
Date ۱۴۰۳/۷/۱۳

«به نام خدا»

فاطمه متقی - ۹

۱)
$$f(f(\tfrac{2}{3})) \Rightarrow f(\tfrac{1}{2}) = \cot\tfrac{\pi}{4}\times\tfrac{2}{3} = \cot\tfrac{\pi}{6} = \sqrt{3}$$
$$f(\sqrt{3}) = \sqrt{(\sqrt{3})^2+1} = \sqrt{3+1} = \sqrt{4} = 2 \Rightarrow fof(\tfrac{2}{3}) = 2$$

۲) الف)
$$f(g(\tfrac{\pi}{3})) \Rightarrow g(\tfrac{\pi}{3}) = 2\cos^2\tfrac{\pi}{3} = 2\times(\tfrac{1}{2})^2 = \tfrac{1}{2}$$
$$f(\tfrac{1}{2}) = \sqrt{\tfrac{2(2)-1}{2^2}} = \sqrt{\tfrac{3}{4}} = \tfrac{\sqrt{3}}{2} \Rightarrow fog(\tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2}$$

ب)
$$f(g(\sqrt{2})) \Rightarrow g(\sqrt{2}) = \tfrac{\sqrt{2}}{1-\sqrt{2}}\times\tfrac{1+\sqrt{2}}{1+\sqrt{2}} = -(\sqrt{2}+2) \simeq -\sqrt{2}-2$$
$$f(-\sqrt{2}-2) = [-\sqrt{2}-2] \simeq [-1/4-2] = [-3/4] = -4$$

۳)
$$g(f(\tfrac{\pi}{4})) \Rightarrow f(\tfrac{\pi}{4}) = \sin\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2}$$
$$g(\tfrac{\sqrt{2}}{2}) = \tfrac{\sqrt{2}}{2}\sqrt{1-(\tfrac{\sqrt{2}}{2})^2} = \tfrac{\sqrt{2}}{2}\sqrt{1-\tfrac{2}{4}} = \tfrac{\sqrt{2}}{2}\times\tfrac{\sqrt{2}}{2} = \tfrac{1}{2}$$

۴)
الف) $fog(x) = \{(4,5),(2,5),(6,12),(8,12)\}$

ب) $gof(x) = \emptyset$

ج) $gog(x) = \{(4,8),(2,6),(6,10)\}$

د) $fof(x) = \emptyset$

۵)
if $a=4 \rightarrow g(4)=3 \rightarrow f(3)=2 \Rightarrow fog=(4,2) \Rightarrow$ ✓

if $b=4 \rightarrow g(4)=1 \rightarrow f(1)=7 \Rightarrow fog=(4,7) \Rightarrow$ ✗

$\rightarrow a=4$ ①

$g(f(4))=1 \rightarrow g(5)=1 \Rightarrow b=5$ ②

①, ② $\Rightarrow (a,b)=(4,5)$

$f(4)=5$

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۶) $f(f(x)) = 4x + 3$ → if $f(x) = ax + b \Rightarrow f(f(x)) = a(ax+b) + b = 4x + 3$

↳ $a^2x + ab + b = 4x + 3 \rightarrow a^2 = 4 \rightarrow a = \pm 2$

if $a = 2 \rightarrow 2b + b = 3 \Rightarrow b = 1 \Rightarrow f(x) = 2x + 1$

if $a = -2 \rightarrow -2b + b = 3 \Rightarrow b = -3 \Rightarrow f(x) = -2x - 3$

$\rightarrow f(-1) = -1$

$g(2x+3) = 3x - 2$ → if $g(x) = cx + d \rightarrow g(2x+3) = c(2x+3) + d = 3x - 2$

$2cx + 3c + d = 3x - 2 \rightarrow 2c = 3 \rightarrow c = \frac{3}{2}$

$d = -2 - \frac{9}{2} = -\frac{13}{2}$

$\rightarrow g(x) = \frac{3}{2}x - \frac{13}{2}$

$g(f(-1)) = g(-1) = -\frac{3}{2} - \frac{13}{2} = -8$

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۷) $g(f(x)) = \frac{1}{(\sqrt{x+|x|})^2 - 4\sqrt{x+|x|}} \rightarrow \frac{1}{\underbrace{|x+|x||}_{\text{همواره نامنفی}} - 4\sqrt{x+|x|}} = \frac{1}{x+|x| - 4\sqrt{x+|x|}}$

↳ $D_{g \circ f} = \mathbb{R} - \{\text{ریشه‌های مخرج}\}$

$x+|x| - 4\sqrt{x+|x|} = 0 \xrightarrow{t = \sqrt{x+|x|}} t^2 - 4t = 0 \begin{cases} t = 4 \Rightarrow x+|x| = 16 \begin{cases} x \le 0 \rightarrow 0 = 16 \times \\ x > 0 \rightarrow x = 8 \end{cases} \\ t = 0 \Rightarrow x \le 0 \end{cases}$

↳ $D_{g \circ f} = (0, +\infty) - \{8\}$

$x \in D_f \Rightarrow x \in \mathbb{R}$

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۸) $(f+g) \circ f \Rightarrow f(f(x)) + g(f(x)) \Rightarrow D_{f(x)} = 1 - x^2 \ge 0 \rightarrow x^2 \le 1 \rightsquigarrow [-1, 1]$

↳ $f(f(x)) = \sqrt{1 - |1 - x^2|} \xrightarrow[\text{تعریف می‌شود}]{\text{در بازه } [-1, 1]} ①$

↳ $g(f(x)) = \sqrt{\sqrt{1 - x^2}} = \sqrt[4]{1 - x^2} \Rightarrow x^2 \le 1 \rightsquigarrow \begin{matrix} x \le 1 \\ x \ge -1 \end{matrix} \rightarrow [-1, 1]$ ②

$f(f(x)) + g(f(x)) \Rightarrow ①, ② \Rightarrow D_{(f+g) \circ f} = [-1, 1]$

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۹) الف) $f\left(\frac{۳x+۱}{x-۲}\right) = ۴x+۵$ $\quad \frac{۳x+۱}{x-۲} = t \to tx - ۲t = ۳x + ۱$

$\hookrightarrow (۳-t)x = -۲t-۱ \to x = \frac{-۲t-۱}{۳-t} = \frac{۲t+۱}{t-۳}$

$f\left(\frac{۲t+۱}{t-۳}\right) = ۴\left(\frac{۲t+۱}{t-۳}\right) + ۵$

$\to f(x) = \frac{۱۳x-۱۱}{x-۳} \Rightarrow (x \neq ۳)$

ب) $f(\overbrace{x+\frac{۱}{x}}^{t}) = x^۳ + \frac{۱}{x^۳} \Rightarrow f(t) = t^۳ - ۳t \Rightarrow f(x) = x^۳ - ۳x$

$(x+\frac{۱}{x})^۳ = x^۳ + \frac{۱}{x^۳} - ۳(x+\frac{۱}{x})$

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۱۰) $g(f(x)) = g(x\sqrt{x}) \xrightarrow{x=۲} g(۲\sqrt{۲})$

$g(x\sqrt{x}) \xrightarrow{x=۱} g(۱) \longrightarrow ۲ - ۱ = \boxed{۱}$